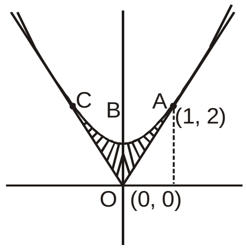
EXERCISE – IV HINTS & SOLUTIONS

Sol.1 $f(x + 1) = f(x) + 2x + 1$; $f(0) = 1$
put $x = 0$
$f(1) = f(0) + 1 = 2$
put $x = 1$
$f(2) = f(1) + 3 = 5$
put $x = 2$
$f(3) = f(2) + 5 = 10$
$\Rightarrow f(x) = 1 + x^2$

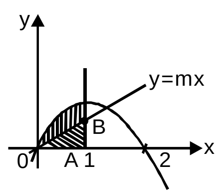


Let the pair of tangents
$y = mx$
$mx = 1 + x^2 \Rightarrow x^2 - mx + 1 = 0$
$D = 0 \Rightarrow m = \pm 2$
pair of tangent $y = \pm 2x$

$$\text{Area} = 2\int_0^1 (1 + x^2 - 2x)\,dx = \frac{2}{3}$$

Sol.2 Area bounded by
$y = 2x - x^2$, $y = 0$ & $x = 1$

$$A = \int_0^1 (2x - x^2)\,dx$$



$$= \frac{2}{3}$$

Let line is $y = mx$ which divides the area into two equal parts so area of Δ OAB $= \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$

$$\Rightarrow \frac{1}{2} \times 1 \times m = \frac{1}{3} \qquad \Rightarrow m = \frac{2}{3}$$

so line is $y = \frac{2}{3}x$

Sol.3 Equation of tangent at (1, 1)
$y - 1 = n(x - 1) \Rightarrow y = nx - n + 1$ ....(i)
area bounded by the curve, the tangent & the x-axis is

$$A = \left|\int_0^1 (nx - n + 1) - x^n).dx\right| = \left|\frac{n^2 - n}{2(n+1)}\right|$$

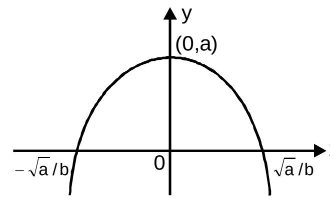
Now A to be maximum, $\frac{dA}{dn} = 0 \Rightarrow$ at $n = \sqrt{2} + 1$

Sol.4 at (2, 1) $\Rightarrow a = 1 + 4b$
area bounded by curve & x-axis

$$A = 2\left[\int_0^{\sqrt{a/b}} ((1+4b) - bx^2)\,dx\right]$$

for area to be minimum $\frac{dA}{db} = 0$

$$\Rightarrow b = \frac{1}{8}$$



$$\& \; a = \frac{3}{2}$$

$A_{min} = 4\sqrt{3}$ sq. units.

Sol.5 From the figure it is clear that

$$\int_0^a (\sin x - f(x))dx = 1 - \sin a + (a - 1)\cos a$$

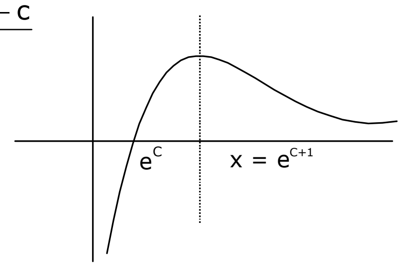
differentiate w.r.t. a
$\sin a - f(a) = -\cos a + \cos a - (a-1)\sin a$
$\Rightarrow \sin a - f(a) = -a \sin a + \sin a$
$\Rightarrow f(a) = a \sin a \Rightarrow f(x) = x \sin x$
The points where f(x) & sinx intersect are
$x \sin x = \sin x \Rightarrow \sin x = 0$ or $x = 1$.
We can say that $a = 1$

$$A_1 = \int_0^1 (\sin x - x\sin x)dx = (1 - \sin 1)$$

$$A_2 = \int_1^{\pi} (f(x) - \sin x)dx = \int_1^{\pi} (x\sin x - \sin x)dx = (\pi - 1 - \sin 1)$$

$$A_3 = \left|\int_{\pi}^{2\pi} (\sin x - x\sin x)dx\right| = (3\pi - 2)$$

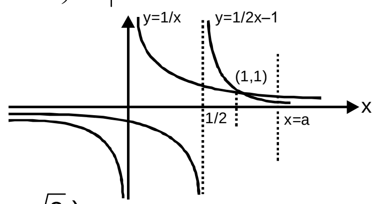
Sol.6 $y = \frac{\ell n\, x - c}{x}$



$\frac{dy}{dx} = 0$
$\Rightarrow x = e^{c+1}$
$y = 0 \Rightarrow x = e^c$

$$\text{Area} = \int_{e^c}^{e^{c+1}} \left(\frac{\ell n\, x - c}{x}\right) dx = \frac{1}{2}$$

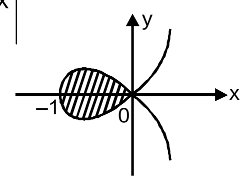
Sol.7 $A = \left|\int_1^a \left(\frac{1}{x} - \frac{1}{2x-1}\right) dx\right|$



$= \ell n\ \frac{4}{\sqrt{5}}$

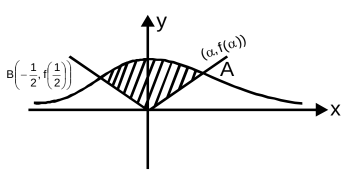
$a = 8$ on $\frac{2}{5}\ (6 - \sqrt{2}\ )$

Sol.8 $\text{Area} = 2\left|\int_1^0 \left(x\sqrt{\frac{1+x}{1-x}}\right) dx\right|$



$= 2 - \frac{\pi}{2}$

Sol.9 $y = \frac{1}{1+x^2}$



$$A = \int_{-1/2}^{0} \left(\frac{1}{1+x^2} - \left(\frac{\alpha^3}{1+\alpha^2}\right)x\right) dx$$

$$+ \int_0^{\alpha} \left(\frac{1}{1+x^2} - \frac{1}{\alpha(1-\alpha^2)}x\right) dx$$

for area to be minimum $\frac{dA}{d\alpha} = 0$ & $A = \frac{\sqrt{1}-1}{2}$

Sol.10 Equation of line

$y - c = m\ (x - 1)$

$\Rightarrow\ y = mx + c - m \qquad$ ....(i)

solving line & parabola

$x = \frac{m \pm \sqrt{m^2 - 4m + 4c}}{2} \Big< {x_1 \atop x_2}$ (let)

$A = \left|\int_{x_1}^{x_2} (x^2 - (mx + c - m))\ dx\right| = 36$

Sol.11 $A_n = \int_0^{\pi/4} (\tan x)^n\ dx$

$0 < \tan x < 1$ when $0 < x < \pi/4$

$0 < (\tan x)^{n+1} < (\tan x)^n,\ n \in N$

$\Rightarrow\ \int_0^{\pi/4} (\tan x)^{n+1} dx < \int_0^{\pi/4} (\tan x)^n\ dx$

$\Rightarrow\ A_{n+1} < A_n$

for $n < 2$

$A_n + A_{n+2} = \int_0^{\pi/4} [(\tan x)^n + (\tan x)^{n+2}]\ dx$

$= \int_0^{\pi/4} (\tan x)^n\ (1 + \tan^2 x)\ dx$

$= \frac{1}{n+1}$

since $A_{n+2} < A_{n+1} < A_n$

so $A_n + A_{n+2} < 2A_n$

$\Rightarrow\ \frac{1}{n+1} < 2A_n\ \Rightarrow\ \frac{1}{2n+2} < A_n \qquad$ .....(1)

Also for $n > 2$; $A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$

$\Rightarrow\ 2A_n < \frac{1}{n-1}\ \Rightarrow\ A_n < \frac{1}{2n-2} \qquad$ .....(2)

combining (1) & (2) use get

$\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$.

Sol.12 Proof. $A = \int_a^c (f(c) - f(x))\ dx + \int_c^b f(x) - f(c))\ dx$

$= f(c)\ [c - a + c - b] - \int_a^c f(x)\,dx + \int_c^b f(x)\ dx$

differentiating w.r.t. 'c'

$\frac{dA}{dc} = f(c).2 + (2c - a - b).\ f'(c) - f(c) - f(b)\ .\ 0 - f(c)$

$\Rightarrow\ (2c - a - b)\ (f'(c)) = 0\ \Rightarrow\ 2c - a - b = 0$

$c = \frac{a+b}{2}$

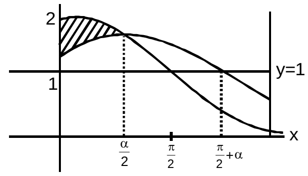
$f'(x) = x^2 - 2x + a = 0$

$\frac{2+a}{2} = -a \qquad \Rightarrow\ a = \frac{2}{3}$

Sol.13 $1 + \cos x = 1 + \cos (x - \alpha)$

$\Rightarrow x = \dfrac{\alpha}{2}$

Now $\alpha/2$



$$\int_0^{\alpha/2} (1+\cos x) - (1+\cos(x-\alpha)))\,dx$$

$$= -\int_{\frac{\pi}{2}+\alpha}^{n} (1-(1+\cos(\pi-\alpha)))\,dx$$

$$\Rightarrow \sin x - \sin(x-a)\Big|_0^{\alpha/2} = \sin(x-\alpha)\Big|_{\pi}^{\frac{\pi}{2}+\alpha}$$

$$\Rightarrow 2\sin\frac{\alpha}{2} - \sin\alpha = 1 - \sin\alpha \Rightarrow \alpha = \frac{\pi}{3}$$

Area bounded by c_1, c_2 & $y = 0$

$$= \int_0^{\pi/6} (c_1 - c_2)\,dx + \int_{\pi/6}^{\pi} (c_2 - c_1)\,dx = 2 \text{ sq. units}$$

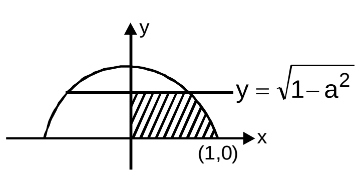
Sol.14 $x^2y^2 = a^2y^2 - a^2x^2$

$$y^2 = \frac{a^2x^2}{a^2 - x^2}$$

$$\text{Area} = 4\int_0^a \frac{ax}{\sqrt{a^2 - x^2}}\,dx \quad = 14a^2$$

Sol.15 $A = \displaystyle\int_0^{\sqrt{1-a^2}} (\sqrt{1-y})\,dy$

$$\frac{dA}{da} = 0,\ a = \frac{1}{2}$$



so $A_{min} = \dfrac{3\sqrt{3} - \pi}{12}$ & maxima at $a = 0$.

Sol.16 Let a point $P(t, t^2)$ is on the curve c.

y cordinate of Q is also t^2 & $R = (t, t^2/2)$

& $Q = (x, t^2)$

Area of OPQ = Ara of OPR

$$\Rightarrow \int_0^{t^2} (\sqrt{y} - f(x)\,dy = \int_0^t (x^2 - x^2/2)\,dx$$

differntiating both sides w.r.t. t

$$f(t) = \frac{16}{9}t^2 \Rightarrow f(x) = \left(\frac{16}{9}\right)x^2$$

Sol.17 $f(x) = \displaystyle\int_0^x e^t \ell n \sec t\,dt - \int_0^x \sec^2 t\,dt$

$$= e^t \ell n \sec t\,\Big|_0^x - \int_0^x e^t \tan t\,dt - \int_0^x \sec^2 t\,dt$$

$$= e^x \ell n \sec x - \int_0^x e^t(\tan t + \sec^2 t)\,dt$$

$$= e^x \ell n \sec x - e^x \tan x$$

$$\text{Area} = \int_0^{\pi/3} (f(x) - g(x))\,dx$$

$$= \int_0^{\pi/3} (e^x \ell n \sec x + e^x \tan x)\,dx$$

$$= e^x \ell n \sec x\,\Big|_0^{\pi/3}$$

$$= e^{\pi/3} \ell n\, 2$$